

Glimpse of Translation – Modulation Invariant Banach Spaces of generalized functions

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dedicated to the 70-th birthday of prof. Stevan Pilipović

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Plan of the talk

- 1 Notation
- 2 TMIB spaces
- 3 Results and generalizations

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(M.6) $p! \subset M_p$, i.e. there exist $c_0, L_0 > 0$ such that $p! \leq c_0 L_0^p M_p$, $p \in \mathbb{N}$.

$M(\cdot)$ and $A(\cdot)$, are the associated functions for M_p and A_p , defined by

$$M(\rho) = \sup_{p \in \mathbb{N}} \ln_+ \frac{\rho^p}{M_p}, \quad A(\rho) = \sup_{p \in \mathbb{N}} \ln_+ \frac{\rho^p}{A_p} \quad \text{for } \rho > 0,$$

respectively.

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$$\mathcal{S}_{(A_p)}^{(M_p)} = \varprojlim_{h \rightarrow \infty} \mathcal{S}_{A_p, h}^{M_p, h}, \quad \mathcal{S}_{\{A_p\}}^{\{M_p\}} = \varinjlim_{h \rightarrow 0} \mathcal{S}_{A_p, h}^{M_p, h}.$$

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$$\sigma_{h,A_p}^{\emptyset,k}(\varphi) = \sup_{\alpha \leq k} \|e^{A(h|\cdot|)} D^\alpha \varphi(\cdot)\|_{L^\infty(\mathbb{R}^n)} \quad (1)$$

$$\sigma_{h,\emptyset,k}^{M_p}(\varphi) = \sup_{\alpha \in \mathbb{N}_0^n} \frac{h^\alpha}{M_\alpha} \|\langle \cdot \rangle^k D^\alpha \varphi(\cdot)\|_{L^\infty(\mathbb{R}^n)} \quad (2)$$

$$\sigma_{\emptyset,h}^{\emptyset,k}(\varphi) = \sup_{\alpha \leq k, \beta \leq h} \|\langle \cdot \rangle^\beta D^\alpha \varphi(\cdot)\|_{L^\infty(\mathbb{R}^n)}. \quad (3)$$

$$\mathcal{S}_{(A_p)}^\emptyset(\mathbb{R}^n) = \varprojlim_{h \rightarrow \infty} \varprojlim_{k \rightarrow \infty} \mathcal{S}_{A_p, h}^{\emptyset, k}, \quad \mathcal{S}_\emptyset^{(M_p)}(\mathbb{R}^n) = \varprojlim_{h \rightarrow \infty} \varprojlim_{k \rightarrow \infty} \mathcal{S}_{h, \emptyset, k}^{M_p}.$$

$$\mathcal{S}_\emptyset^\emptyset(\mathbb{R}^n) = \varprojlim_{h \rightarrow \infty} \varprojlim_{k \rightarrow \infty} \mathcal{S}_{\emptyset, h}^{\emptyset, k}.$$

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" \hookrightarrow " stands for dense and continuous linear embedding between t.v.s.

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Translation-modulation invariant Banach spaces of tempered ultradistributions

TMIB

A (B) -space E is said to be a translation-modulation invariant (B) -space of ultradistributions (in short: TMIB)) of class $* - \dagger$ if it satisfies the following three conditions:

- (a) The continuous and dense inclusions $\mathcal{S}_\dagger^*(\mathbb{R}^d) \hookrightarrow E \hookrightarrow \mathcal{S}'_\dagger^*(\mathbb{R}^d)$ hold.
- (b) $T_x(E) \subseteq E$ and $M_\xi(E) \subseteq E$ for all $x, \xi \in \mathbb{R}^d$.
- (c) There exist $\tau, C > 0$ (for every $\tau > 0$ there exists $C_\tau > 0$), such that

$$\begin{aligned}\omega_E(x) &:= \|T_x\|_{\mathcal{L}_b(E)} \leq Ce^{M(\tau|x|)} \quad \text{and} \\ \nu_E(\xi) &:= \|M_{-\xi}\|_{\mathcal{L}_b(E)} \leq Ce^{M(\tau|\xi|)}. \end{aligned}\tag{4}$$

The functions $\omega_E : \mathbb{R}^d \rightarrow (0, \infty)$ and $\nu_E : \mathbb{R}^d \rightarrow (0, \infty)$ defined in (4) are called the **weight functions** of the translation and modulation groups of E , respectively (in short: its weight functions).

Such space, in $\emptyset - \emptyset$, $M_p - \emptyset$ or $\emptyset - A_p$ case, is called a **TMIB space of distributions**, and in the other cases **TMIB space of ultradistributions**, of $* - \dagger$ type respectively.

A space is a **DTMIB of ultradistributions** of class $* - \dagger$ if it is the strong dual of a **TMIB** of class $* - \dagger$.

For $* = \emptyset$ or $\dagger = \emptyset$ the associated functions are $M(\rho) = \ln(1 + |\rho|)$ or $A(\rho) = \ln(1 + |\rho|)$, respectively.

Let E be a TMIB of class $* - \dagger$.

The convolution and multiplication

$$*: \mathcal{S}_\dagger^*(\mathbb{R}^d) \times \mathcal{S}_\dagger^*(\mathbb{R}^d) \rightarrow \mathcal{S}_\dagger^*(\mathbb{R}^d),$$

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E becomes a Banach module over $L_{\omega_E}^1$ and A_{ν_E} with respect to these operations,

$$\|g * f\|_E \leq \|g\|_{L_{\omega_E}^1} \|f\|_E, \quad g \in L_{\omega_E}^1, \quad \forall f \in E, \quad (5)$$

and

$$\|h \cdot f\|_E \leq \|h\|_{A_{\nu_E}} \|f\|_E, \quad \forall h \in A_{\nu_E}, \quad \forall f \in E. \quad (6)$$



Furthermore, the convolution of $f \in E$ and $g \in L^1_{\omega_E}$ can be represented as a Bochner integral of an E -valued function,

$$g * f = \int_{\mathbb{R}^d} g(y) T_y f dy, \quad (7)$$

while its multiplication with $h \in A_{\nu_E}$ is given by the Bochner integral

$$h \cdot f = \int_{\mathbb{R}^d} (\mathcal{F}^{-1} h)(\xi) M_{-\xi} f d\xi. \quad (8)$$

Let E and F be TMIB of class $* - \dagger$ on \mathbb{R}^{d_1} and \mathbb{R}^{d_2} . Then,

- (i) $E \hat{\otimes}_\epsilon F$ is a TMIB of class $* - \dagger$ on $\mathbb{R}^{d_1+d_2}$ and
 $\omega_{E \hat{\otimes}_\epsilon F} = \omega_E \otimes \omega_F$ and $\nu_{E \hat{\otimes}_\epsilon F} = \nu_E \otimes \nu_F$.
- (ii) If either E or F satisfies the approximation property, then
 $E \hat{\otimes}_\pi F$ is also a TMIB class $* - \dagger$ with $\omega_{E \hat{\otimes}_\pi F} = \omega_{E \hat{\otimes}_\epsilon F}$ and
 $\nu_{E \hat{\otimes}_\pi F} = \nu_{E \hat{\otimes}_\epsilon F}$.

Let $g \in \mathcal{S}^*(\mathbb{R}^n) \setminus \{0\}$ and $f \in \mathcal{S}'^*(\mathbb{R}^n)$.

Short-time Fourier transform of f with respect to a window function g is

$$V_g f(x, \xi) = \mathcal{F}_{t \rightarrow \xi} ((\overline{T_x g} f)(t))$$

for all $x, \xi \in \mathbb{R}^n$.

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$$\mathcal{M}^F = \{f \in \mathcal{S}'^*(\mathbb{R}^n) \mid V_g f \in F\}$$

provided with the norm $\|\cdot\|_{\mathcal{M}^F} = \|V_g(\cdot)\|_F$.

Let F be either a $TMIB$ or a $DTMIB$ of class $*$ on \mathbb{R}^{2n} .

$$\tilde{\omega}_F(x) = \omega_F(x, 0)\nu_F(0, x) \text{ and } \tilde{\nu}(x) = \omega_F(0, -x), x \in \mathbb{R}^n.$$

Let F be a Banach space of ultradistributions on \mathbb{R}^{2n} .

(i) If F is a $TMIB$ of class $*$, then

$$\mathcal{M}_{\tilde{\omega}_F \otimes \tilde{\nu}}^{L^1} \hookrightarrow \mathcal{M}^F.$$

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(iii) If F is a reflexive *TMIB* of class $*$, then

$$\mathcal{M}_{\tilde{\omega}_F \otimes \tilde{\nu}}^{L^1} \hookrightarrow \mathcal{M}^F \rightarrow \mathcal{M}_{1/\check{\omega} \otimes 1/\check{\nu}}^{L^\infty}.$$

Examples

For $1 \leq p, q < \infty$, the weighted mixed-norm space $L_\omega^{p,q}(\mathbb{R}^{2n})$,

$$\|f\|_{L_\omega^{p,q}} = \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x, \xi)|^p \omega(x, \xi)^p dx \right)^{q/p} d\xi \right)^{1/q} < \infty,$$

are *TMIB* spaces.

Let $F = L_\omega^{p,q}(\mathbb{R}^{2n})$. Then the spaces \mathcal{M}^F are the standard modulation spaces of (ultra)distributions.

When $1 < p \leq 2$, $L^p(\mathbb{R}^n) \hat{\otimes}_\pi L^p(\mathbb{R}^n)$ is not a solid Banach space.

The spaces $L_{\omega_1}^2 \hat{\otimes}_\pi L_{\omega_2}^2$ and $L_{\omega_1}^2 \hat{\otimes}_\epsilon L_{\omega_2}^2$ are TMIB spaces different from $L_{\omega_1 \otimes \omega_2}^{p,q}(\mathbb{R}^{2n})$ for $1 \leq p, q \leq \infty$.

If $M_p/p!$ and $A_p/p!$ satisfy condition (M.1), i.e condition (M.4) then there are TMIB of class $* - \dagger$ that exist arbitrarily close to $\mathcal{S}_\dagger^*(\mathbb{R}^d)$ and $\mathcal{S}'_\dagger^*(\mathbb{R}^d)$

The wave front set with respect to a TMIB space

Let M_p , $p \in \mathbb{N}$, be a sequence of positive numbers satisfying $M_0 = M_1 = 1$ also (M_1) , (M_2) , (M_4) and additionally

(M.3) there exists $c_0 \geq 1$ such that

$$\sum_{q=p+1}^{\infty} M_{q-1}/M_q \leq c_0 p M_p/M_{p+1}, p \in \mathbb{Z}_+.$$

Let E be TMIB space of class $*$ over \mathbb{R}^d . Besides the properties (a), (b) and (c) of the Definition of TMIB space and additionally

- (d)** $\mathcal{F}E$ is a solid space, i.e. $\mathcal{F}E \subseteq L^1_{\text{loc}}(\mathbb{R}^d)$ and there exists $C_0 > 0$ such that if $g \in L^1_{\text{loc}}(\mathbb{R}^d)$, $f \in \mathcal{F}E$ and $|g(x)| \leq |f(x)|$ a.e. then $g \in \mathcal{F}E$ and $\|g\|_{\mathcal{F}E} \leq C_0 \|f\|_{\mathcal{F}E}$.

For $f \in \mathcal{E}'^*(\mathbb{R}^d)$ we define the set $\Sigma_E(f) \subseteq \mathbb{R}^d \setminus \{0\}$ as follows:

$\xi \in \mathbb{R}^d \setminus \{0\}$ does not belong to $\Sigma_E(f)$ if and only if there exists a cone neighbourhood Γ of ξ such that

$$\theta_\Gamma \mathcal{F}f \in \mathcal{F}E, \quad (9)$$

where θ_Γ denotes the characteristic function of Γ .

For $f \in \mathcal{D}'^*(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, we define

$$\Sigma_{x,E}(f) = \bigcap_{\chi \in \mathcal{D}^*(\mathbb{R}^d), \chi(x) \neq 0} \Sigma_E(\chi f).$$

Let $\psi \in \mathcal{D}^*(\mathbb{R}^d)$ and $f \in \mathcal{E}'^*(\mathbb{R}^d)$. Then $\Sigma_E(\psi f) \subseteq \Sigma_E(f)$.

Let $f \in \mathcal{D}'^*(\mathbb{R}^d)$, $x \in \mathbb{R}^d$ and Γ be an open cone such that $\Sigma_{x,E}(f) \subseteq \Gamma$.

There exists $\chi \in \mathcal{D}^*(\mathbb{R}^d)$ satisfying $\chi(x) \neq 0$ and having support arbitrarily close to x such that $\Sigma_E(\chi f) \subseteq \Gamma$. In particular, $\Sigma_{x,E}(f) = \emptyset$ if and only if there exists $\chi \in \mathcal{D}^*(\mathbb{R}^d)$, satisfying $\chi(x) \neq 0$, such that $\chi f \in E$.

WFS

For $f \in \mathcal{D}'^*(\mathbb{R}^d)$, we define the **E -wave front set** of f by

$$WF_E(f) = \{(x, \xi) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\}) \mid \xi \in \Sigma_{x,E}(f)\}.$$

For $f \in \mathcal{D}'^*(\mathbb{R}^d)$, we can define the set **sing supp _{E} f** $\subseteq \mathbb{R}^d$ whose complement is given by the points at which f locally behaves as an element of E .

Remark

When E is a Sobolev space $H^s(\mathbb{R}^d)$, $s \in \mathbb{R}$, and $f \in \mathcal{D}'(\mathbb{R}^d)$, then the definition of $WF_E(f)$ coincides with the Sobolev wave front set of f as defined by Hörmander.

Theorem

Let $f \in \mathcal{D}'^*(\mathbb{R}^d)$ and $(x_0, \xi_0) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})$. The following conditions are equivalent.

- (i) $(x_0, \xi_0) \notin WF_E(f)$.
- (ii) There exist a cone neighbourhood Γ of ξ_0 , a compact neighbourhood K of x_0 and $\chi \in \mathcal{D}^*(\mathbb{R}^d)$, satisfying $\chi(0) \neq 0$ such that

$$\theta_\Gamma V_\chi f(x, \cdot) \in \mathcal{F}E, \quad \forall x \in K,$$

and

$$\sup_{x \in K} \|\theta_\Gamma V_\chi f(x, \cdot)\|_{\mathcal{F}E} < \infty.$$

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Thank you for Your attention